

INSTABILITY FOR THE ROTATION SET OF DIFFEOMORPHISMS OF THE TORUS HOMOTOPIC TO THE IDENTITY

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ABSTRACT. The aim of this short note is to explain how the arguments of the “closing lemma with time control” of F. Abdenur and S. Crovisier [AC12] can be used to answer Question 1 of the article “Instability for the rotation set of homeomorphisms of the torus homotopic to the identity” of S. Addas-Zanata [AZ04].

In this short note, we explain how to get a C^1 version of a perturbation result of the rotation set of homeomorphisms of the torus homotopic to the identity, obtained by S. Addas-Zanata in [AZ04]: consider some diffeomorphism f of the torus, isotopic to the identity, and suppose that some extreme point (t, ω) of the rotation set of f has at least one irrational coordinate. Then there exists a perturbation g of f , which is arbitrarily C^1 -close to f , such that the rotation set of g contains some vector that was not in the rotation set of f .

We will use the notations of [AZ04]. Let us recall the most useful ones: we will denote $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ the flat torus. The space $D^1(\mathbf{T}^2)$ will be the set of C^1 -diffeomorphism of the torus \mathbf{T}^2 homotopic to the identity, endowed with the classical C^1 topology on compact spaces; $D^1(\mathbf{R}^2)$ will be the set of lifts to the plane of elements of $D^1(\mathbf{T}^2)$. Given $\tilde{f} \in D^1(\mathbf{R}^2)$, its *rotation set* will be defied as

$$\rho(\tilde{f}) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n \geq i} \left\{ \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} \mid \tilde{x} \in \mathbf{R}^2 \right\}}.$$

For $\tilde{x} \in \mathbf{R}^2$, we will denote

$$\rho(\tilde{x}, n, \tilde{f}) = \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n}$$

the rotation vector of the segment of orbit $\tilde{x}, \tilde{f}(\tilde{x}), \dots, \tilde{f}^n(\tilde{x})$, and when it is well defined (for example for a periodic point),

$$\rho(\tilde{x}, \tilde{f}) = \lim_{n \rightarrow +\infty} \rho(\tilde{x}, n, \tilde{f}).$$

We will also consider ω a volume or a symplectic form on \mathbf{T}^2 , whose lift to \mathbf{R}^2 will also be denoted by ω .

We will prove the following result.

Theorem 1. *Let $\tilde{f} \in D^1(\mathbf{R}^2)$ be such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin \mathbf{Q}^2$. Then there exists $\tilde{g} \in D^1(\mathbf{R}^2)$, arbitrarily C^1 -close to \tilde{f} , such that $\rho(\tilde{g}) \cap \rho(\tilde{f})^{\mathbb{C}} \neq \emptyset$ (and in particular, $\rho(\tilde{g}) \neq \rho(\tilde{f})$).*

Moreover, if \tilde{f} preserves ω , then \tilde{g} can be supposed to preserve it too.

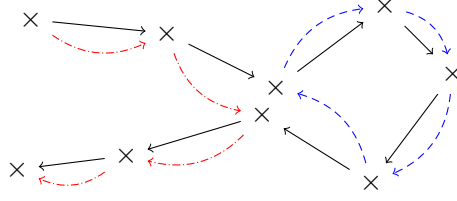


FIGURE 1. If the rotation vector of the initial orbit (in black) is in $\{L > 0\}$, then the rotation vector of one of the two pseudo orbits (in red and in blue) too.

We will prove this theorem by replacing the C^0 perturbation result of [AZ04] by a closing lemma in topology C^1 , obtained by adapting the arguments of Theorem 6 of [AC12].

Lemma 2 (Closing lemma with rotation control). *Let $\tilde{f} \in D^1(\mathbf{T}^2)$, $L : \mathbf{R}^2 \rightarrow \mathbf{R}$ a non-trivial affine form, and \mathcal{V} a C^1 -neighbourhood of f . Then, there exists $N \in \mathbf{N}$ such that for every non-periodic point x of f , there exists a neighbourhood V of x such that if $n \geq N$ and $y \in V$ are such that $f^n(y) \in V$ and $L(\rho(\tilde{y}, n, \tilde{f})) > 0$, then there exists $g \in \mathcal{V}$ such that y is a periodic point¹ of g satisfying $L(\rho(\tilde{y}, \tilde{g})) > 0$. Moreover, if f preserves ω , then g can be supposed to preserve it too.*

The idea of the proof of this lemma is identical to that of Theorem 6 of [AC12], by replacing the dichotomy “ ℓ divides / does not divide the length of the orbit” by the dichotomy “ $L(\rho(\tilde{x}, n, \tilde{f})) > 0$ / $L(\rho(\tilde{x}, n, \tilde{f})) \leq 0$ ”. More precisely, the proof of the connecting lemma of S. Hayashi [Hay97] builds a “closable” pseudo-orbit² from a recurrent orbit of f , by making *shortcuts* in this orbit; each time such a shortcut is performed there are two possibilities of creating a new pseudo-orbit (see Figure 1). If the initial orbit belongs to the set $\{L(\rho) > 0\}$, then at least one of these two new pseudo-orbits also belongs to the set $\{L(\rho) > 0\}$ (as the rotation vector of the initial orbit is a barycentre of the two new ones)³.

Proof of Lemma 2. Simply remark that Proposition 4 of [AC12] still holds when condition

3. The length of the periodic pseudo-orbit $(y_1, \dots, y_n = y_0)$ is not a multiple of ℓ .

is replaced by the condition

3. The periodic pseudo-orbit⁴ $(y_1, \dots, y_n = y_0)$ satisfies $L(\frac{\tilde{y}_n - \tilde{y}_0}{n}) > 0$.

The rest of the proof is identical to Section 3.3.1 of [AC12]. \square

We now explain how this connecting lemma with rotation control can be applied to adapt the proof of Theorem 1 of [AZ04] to the C^1 case. Let us quickly recall the main arguments of the proof in the C^0 case. As the rotation set is convex [MZ89], there exists a supporting line of $\rho(\tilde{f})$ at (t, ω) , in other words an affine

¹Note that in general, this period is different from n .

²A pseudo-orbit is called *closable* if Pugh’s algebraic lemma (Lemma 4 of [AC12], see also [Pug67]) can be applied simultaneously to every jump of the pseudo-orbit, to make it become a real orbit.

³This corresponds to the initial argument of [AC12]: “If ℓ does not divide the length of the initial orbit, then it also does not divide the length of at least one of these two new pseudo-orbits”.

⁴To be rigorous here, pseudo-orbits must be considered in the cover \mathbf{R}^2 and perturbations of diffeomorphisms performed in \mathbf{T}^2 .

map $L : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that $L(t, \omega) = 0$ and $L(v) \leq 0$ for every $v \in \rho(\tilde{f})$. Thus, if we build g close to f such that there exists $v \in \rho(\tilde{g})$ satisfying $L(v) > 0$, then we are done.

The ergodic theorem implies the existence of a point $x_0 \in \mathbf{T}^2$ which is recurrent for f and such that $\rho(\tilde{x}_0, \tilde{f}) = (t, \omega)$. At this point there are two possibilities. Either there exists n arbitrarily large such that $f^n(x_0)$ is close to x_0 and $L(\rho(\tilde{x}_0, n, \tilde{f})) > 0$; in this case it suffices to apply a C^0 closing lemma to x_0 and $f^n(x_0)$ to get the theorem. Or for every n large enough such that $f^n(x_0)$ is close to x , we have $L(\rho(\tilde{x}_0, n, \tilde{f})) \leq 0$. This case is a bit more complicated: we begin by proving that in this case, it is possible to suppose that $L(\rho(\tilde{x}_0, n, \tilde{f})) < 0$ (Lemma 3 of [AZ04]). Let n_0 be such a number (large enough); a theorem of recurrence of G. Atkinson [Atk76] implies the existence of a time $n_1 \gg n_0$ such that $L(n_1 \rho(\tilde{x}_0, n_1, \tilde{f}))$ is arbitrarily close to 0. A calculation shows that in this case, $L(\rho(\tilde{f}^{n_0}(\tilde{x}_0), n_1 - n_0, \tilde{f})) > 0$: the rotation vector of the segment of orbit between $\tilde{f}^{n_0}(\tilde{x}_0)$ and $\tilde{f}^{n_1}(\tilde{x}_0)$ belongs to $\{L > 0\}$. It then suffices to apply the C^0 closing lemma to $\tilde{f}^{n_0}(\tilde{x}_0)$ and $\tilde{f}^{n_1}(\tilde{x}_0)$.

Proof of Theorem 1. Let $\tilde{f} \in D^1(\mathbf{R}^2)$ be such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin \mathbf{Q}^2$, and \mathcal{V} a C^1 -neighbourhood of f . We fix once for all a lift \tilde{f} of f , and choose $L : \mathbf{R}^2 \rightarrow \mathbf{R}$ an affine form such that $L(t, \omega) = 0$ and $L(v) \leq 0$ for every $v \in \rho(\tilde{f})$. Let $x_0 \in \mathbf{T}^2$ be a recurrent point of f such that $\rho(\tilde{x}_0, \tilde{f}) = (t, \omega)$. Lemma 2 gives us a number $N \in \mathbf{N}$ and a neighbourhood V of x_0 . The proof of Theorem 1 of [AZ04] summarized in the previous discussion gives us a point $y = f^{n_0}(x_0)$ (with n_0 possibly equal to 0) and a time $n_1 \geq N$ such that $\tilde{f}^{n_1}(\tilde{y}) \in V$ and $L(\rho(\tilde{y}, n_1, \tilde{f})) > 0$. Applying Lemma 2, we get $g \in \mathcal{V}$ such that y is a periodic point of g satisfying $L(\rho(\tilde{y}, \tilde{g})) > 0$. This proves the theorem.

Moreover, if f preserves ω , then g can be supposed to preserve it too. \square

Remark 3. Theorem 1 of [AZ04] is also true in the C^0 measure-preserving case. To see it, it suffices to replace the C^0 closing lemma by the measure-preserving one (see for example Lemma 13 of [OU41]).

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